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1. Introduction

The parabolic equation, originally introduced by Fock and Leontovich (ref. 1), has found successful application in several subject areas that deal with wave propagation issues. In the case of stochastic electromagnetic wave propagation through random media, the use of the parabolic equation was adopted by Klyatskin and Tatarskii 36 years ago (refs. 2 and 3) and has since found much success in describing propagation phenomena from laser beams to optical images. This type of parabolic equation is accurate within the paraxial approximation and is therefore suited to treat small-angle wave scattering about the preferential direction of propagation. The paraxial approximation holds when the wavelength λ and the characteristic size l_0 of the smallest inhomogeneity of the random medium is such that $\lambda \ll l_0$. However, as shown by Klyatskin and Tatarskii, (ref. 4), this parabolic equation in the paraxial approximation is sufficient in describing optical propagation within the Markov approximation (i.e., where the inhomogeneities which compose the random medium are taken to have a δ -function correlation in the direction of wave propagation). There are cases, however, in which the wavelength may be on the order of the size of the scattering inhomogeneities $\lambda \sim l_0$ thus potentially violating the paraxial approximation and requiring an 'extended' parabolic equation that still possess a preferential direction of propagation but is not bound by the paraxial approximation, i.e., can describe scattering at large angles out of the direction of propagation. Such an extended equation can only be derived from the fundamental Helmholtz equation.

It is the purpose of this work to analytically derive solutions for the generalized mutual coherence function (MCF), i.e., the second order moment, of a random wave field propagating through a random medium within the context of the extended parabolic equation. Here, 'generalized' connotes the consideration of both the transverse as well as the longitudinal second order moments (with respect to the direction of propagation). Such solutions will afford a comparison between the results of the parabolic equation within the paraxial approximation and those of the wide-angle extended theory. In section 2, the extended parabolic equation for electromagnetic wave propagation is given which, of course, derives from the stochastic Helmholtz equation. Since this will be an operator equation in the random electric field, a statistical operator method is developed in section 3 which will give a general equation for an arbitrary spatial statistical moment of the wave field. The generality of the operator method allows one to obtain an expression for the second order field moment in the direction longitudinal to the direction of propagation. From this, expressions are obtained as a special case for the MCF of the field, both in the transverse and longitudinal directions. Analytical solutions to these equations are derived for the Kolmogorov and Tatarskii spectra of atmospheric permittivity fluctuations within the Markov approximation.

2. The Wide-Angle Extended Parabolic Equation

Consider the scalar stochastic Helmholtz equation for an electric field $E(x, \vec{\rho})$ propagating principally along an x-axis and perpendicular to the $\vec{\rho}$ -plane of an otherwise arbitrary coordinate system

$$\frac{\partial^2 E(x, \vec{\rho})}{\partial x^2} + \nabla_{\vec{\rho}}^2 E(x, \vec{\rho}) + k^2 E(x, \vec{\rho}) = k^2 \tilde{\epsilon}(x, \vec{\rho}) E(x, \vec{\rho}) \quad (1)$$

where $\tilde{\epsilon}(x, \vec{\rho})$ is the random part of the total permittivity $\epsilon(x, \vec{\rho}) = 1 + \tilde{\epsilon}(x, \vec{\rho})$ of the propagation medium.

Using the decomposition of the total field into a forward propagating field $E^+(x, \vec{\rho})$ and a backward propagating field $E^-(x, \vec{\rho})$, i.e.,

$$E(x, \vec{\rho}) = E^+(x, \vec{\rho}) + E^-(x, \vec{\rho}), \quad \frac{\partial E(x, \vec{\rho})}{\partial x} = \frac{\partial E^+(x, \vec{\rho})}{\partial x} + \frac{\partial E^-(x, \vec{\rho})}{\partial x} \quad (2)$$

with the expansion into inhomogeneous plane waves

$$E^{\pm}(x, \vec{\rho}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\pm i(\vec{q} \cdot \vec{\rho} \pm i(k^2 - q^2)^{1/2} x)} d^2 q \quad (3)$$

one can obtain the following operator expressions for the component fields

$$2i \frac{\partial E^{\pm}(x, \vec{\rho})}{\partial x} \pm 2(k^2 + \nabla_{\vec{\rho}}^2)^{1/2} E^{\pm}(x, \vec{\rho}) = \pm k^2 (k^2 + \nabla_{\vec{\rho}}^2)^{-1/2} \tilde{\epsilon}(x, \vec{\rho}) [E^+(x, \vec{\rho}) + E^-(x, \vec{\rho})] \quad (4)$$

such an equation (or its Fourier transform) was considered by Malakov and Saichev (ref. 5) as well as by Klyatskin (ref. 6, p. 169) and Frankenthal and Beran (ref. 7).

In the event that the backscattered field is insignificant with respect to the forward scattered field, i.e., $E^+(x, \vec{\rho}) \gg E^-(x, \vec{\rho})$, one can formally set $E^-(x, \vec{\rho}) = 0$; equation (4) thus becomes a single equation for the forward propagating field

$$2i \frac{\partial E^+(x, \vec{\rho})}{\partial x} + 2(k^2 + \nabla_{\vec{\rho}}^2)^{1/2} E^+(x, \vec{\rho}) = k^2 (k^2 + \nabla_{\vec{\rho}}^2)^{-1/2} \tilde{\epsilon}(x, \vec{\rho}) E^+(x, \vec{\rho}) \quad (5)$$

This equation, which can be called the “extended parabolic equation” (ref. 2, p. 169), was previously analyzed by Saichev (ref. 8) using a method different from the operator formulation in what is to follow. Its integral formulation is also known as the method of multiple forward scatter (refs. 9 and 10). However, in (ref. 10) it was pointed out that this method is applicable to situations where the wavelength λ and the smallest scale of inhomogeneity l_0 is such that $\lambda < l_0$ (ref. 11); The classical parabolic equation in the paraxial approximation holds for cases where $\lambda \ll l_0$. Hence, although equation (5) is capable of describing propagation situations in which the wave is scattered at angles up to $\pi/2$ with respect to the x-axis, its application is limited to cases where $\lambda < l_0$.

The form of equation (5) can be simplified by defining the differential operator

$$U_{\rho} \equiv 2k^2 \left(1 + \frac{\nabla_{\vec{\rho}}^2}{k^2} \right)^{1/2} \quad (6)$$

and the corresponding integral operator

$$V_{\rho} \equiv \frac{1}{2k^2} \left(1 + \frac{\nabla_{\vec{\rho}}^2}{k^2} \right)^{-1/2} \quad (7)$$

Equation (5) then becomes

$$2ik \frac{\partial E(x, \vec{\rho})}{\partial x} + U_{\rho} E(x, \vec{\rho}) - 2k^4 V_{\rho} \tilde{\epsilon}(x, \vec{\rho}) E(x, \vec{\rho}) = 0 \quad (8)$$

Hereafter, the superscript ‘+’ will be dropped. This equation can be reduced to the well-known stochastic parabolic equation in the paraxial approximation by expanding the operators to give $U_{\rho} \approx 2k^2 + \nabla_{\vec{\rho}}^2$ and $V_{\rho} \approx 1/2k^2$ and transforming the field via $E(x, \vec{\rho}) = W(x, \vec{\rho}) \exp(ikx)$. In what is to follow, this relation will be employed to yield an operator equation for the generalized statistical moments of the field. Here, the term ‘generalized’ connotes moments at differing transverse coordinates $\vec{\rho}$ as well as differing longitudinal coordinates x .

3. Operator Solutions for the Generalized Field Moments

3.1 An Expression for the Generalized nm^{th} Field Moments

Defining the stochastic operator

$$D_{x,\rho} \equiv 2ik \frac{\partial}{\partial x} - U_{\rho} + 2k^4 V_{\rho} \tilde{\epsilon}(x, \vec{\rho}) \quad (9)$$

equation (8) simply becomes

$$D_{x,\rho} E(x, \vec{\rho}) = 0 \quad (10)$$

which is easily amenable to further statistical analysis. To this end, one defines the generalized moment of the electric field

$$\Gamma_{nm}(x_1, \vec{\rho}_1; x_2, \vec{\rho}_2; \dots x_n, \vec{\rho}_n; x_{n+1}, \vec{\rho}_{n+1}; \dots x_{n+m}, \vec{\rho}_{n+m}) \equiv \langle g_{nm} \rangle \quad (11)$$

$$g_{nm} \equiv \prod_{j=1}^n E(x_j, \vec{\rho}_j) \prod_{l=n+1}^{n+m} E^*(x_l, \vec{\rho}_l) \quad (12)$$

Using a modification of the prescription set fourth in, e.g., (refs. 12 and 13), one can employ equation (10) for each of the field points $(x, \vec{\rho})$ and obtain for the product of fields

$$L_{nm}g_{nm}(x_1, \vec{\rho}_1; x_2, \vec{\rho}_2; \cdots x_n, \vec{\rho}_n; x_{n+1}, \vec{\rho}_{n+1}; \cdots x_{n+m}, \vec{\rho}_{n+m}) = 0 \quad (13)$$

where

$$\begin{aligned} L_{nm} \equiv & 2ik \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} + \sum_{l=n+1}^{n+m} \frac{\partial}{\partial x_l} \right) + \sum_{j=1}^n (U_{\rho_j} + 2k^4 V_{\rho_j} \tilde{\epsilon}(x_j, \rho_j)) - \\ & - \sum_{l=n+1}^{n+m} (U_{\rho_l}^* + 2k^4 V_{\rho_l}^* \tilde{\epsilon}^*(x_l, \rho_l)) \end{aligned} \quad (14)$$

In order to isolate the quantity $\langle g_{nm} \rangle = \Gamma_{nm}$ from this relation, it is expedient to adopt the methods of (refs. 14 and 15) and decompose the operator L_{nm} into its average and random parts, i.e.,

$$\langle L_{nm} \rangle \equiv 2ik \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} + \sum_{l=n+1}^{n+m} \frac{\partial}{\partial x_l} \right) + \sum_{j=1}^n U_{\rho_j} - \sum_{l=n+1}^{n+m} U_{\rho_l}^* \quad (15)$$

and

$$\tilde{L}_{nm} \equiv 2k^4 \left[\sum_{j=1}^n V_{\rho_j} \tilde{\epsilon}(x_j, \rho_j) - \sum_{l=n+1}^{n+m} V_{\rho_l}^* \tilde{\epsilon}^*(x_l, \rho_l) \right], \quad \langle \tilde{L}_{nm} \rangle = 0 \quad (16)$$

Hence, equation (13) becomes

$$(\langle L_{nm} \rangle + \tilde{L}_{nm})g_{nm} = 0 \quad (17)$$

Ensemble averaging this relation yields

$$\langle L_{nm} \rangle \Gamma_{nm} + \langle \tilde{L}_{nm} g_{nm} \rangle = 0 \quad (18)$$

Similarly writing

$$g_{nm} = \Gamma_{nm} + \tilde{g}_{nm}, \quad \langle \tilde{g}_{nm} \rangle = 0 \quad (19)$$

and substituting into equation (18) gives

$$\langle L_{nm} \rangle_{nm} \langle \tilde{L}_{nm} \tilde{g}_{nm} \rangle = 0 \quad (20)$$

Remembering that it is the goal of this development to obtain an expression for the general field moment Γ_{nm} , one follows the development given in reference 15 and subtracts equation (20) from equation (13) and using equation (19) obtains

$$L_{nm}\Gamma_{nm} + L_{nm}\tilde{g}_{nm} - \langle L_{nm} \rangle \Gamma_{nm} - \langle \tilde{L}_{nm} \tilde{g}_{nm} \rangle = 0 \quad (21)$$

Combining the first and third members of this equation using the fact that $[L_{nm} - \langle L_{nm} \rangle] \Gamma_{nm} = \tilde{L}_{nm} \Gamma_{nm}$ gives

$$L_{nm}\tilde{g}_{nm} + \tilde{L}_{nm}\Gamma_{nm} - \langle \tilde{L}_{nm} \tilde{g}_{nm} \rangle = 0 \quad (22)$$

One must now isolate the random quantity \tilde{g}_{nm} by defining an operator L_{nm}^{-1} inverse to L_{nm} , i.e., $L_{nm}^{-1} L_{nm} = 1$. Thus, operating on equation (22) with L_{nm}^{-1} yields

$$\tilde{g}_{nm} + L_{nm}^{-1} \tilde{L}_{nm} \Gamma_{nm} - L_{nm}^{-1} \langle \tilde{L}_{nm} \tilde{g}_{nm} \rangle = 0 \quad (23)$$

Finally, operating on this relation with \tilde{L}_{nm} , ensemble averaging and solving the resulting expression for $\langle \tilde{L}_{nm} \tilde{g}_{nm} \rangle$ gives

$$\langle \tilde{L}_{nm} \tilde{g}_{nm} \rangle = -[1 - \langle \tilde{L}_{nm} L_{nm}^{-1} \rangle]^{-1} \langle \tilde{L}_{nm} L_{nm}^{-1} \tilde{L}_{nm} \rangle \Gamma_{nm} \quad (24)$$

Substituting this result back into equation (20), one obtains for the equation governing Γ_{nm}

$$\left\{ \langle L_{nm} \rangle - [1 - \langle \tilde{L}_{nm} L_{nm}^{-1} \rangle]^{-1} \langle \tilde{L}_{nm} L_{nm}^{-1} \tilde{L}_{nm} \rangle \right\} \Gamma_{nm} = 0 \quad (25)$$

The solution of this operator equation gives an exact solution for the arbitrary field moments for wide-angle propagation through a random medium characterized by the stochastic permittivity $\tilde{\epsilon}(x, \vec{\rho})$ and the assumption that $\langle \tilde{L}_{nm} \rangle = 0$. Just as in the case of the application of the local method of small perturbations (ref. 12), an explicit assumption governing the statistics of the fluctuations $\tilde{\epsilon}(x, \vec{\rho})$ does not need to be made in the evaluation of the ensemble averages, as will now be demonstrated.

The general relation given by equation (25) can be reduced to the parabolic equation for the field moments in the paraxial approximation in the case where $\lambda \ll l$. In this instance, one employs the approximations for the operators $U_{\rho} \approx 2k^2 + \nabla_{\rho}^2$ and $V_{\rho} \approx 1/2k^2$ used earlier. In addition, the classical parabolic equation considers statistical moments in the same transverse plane, i.e., $x_j = x_l = x$. Thus, the partial differential operators in equation (15) collapse into the single operator $\partial/\partial x$. Equations (15) and (16) then become

$$\langle L_{nm} \rangle \approx 2ik \frac{\partial}{\partial x} + \sum_{j=1}^n \nabla_{\rho_j}^2 - \sum_{l=1+n}^{n+m} \nabla_{\rho_l}^2 \quad (26)$$

and

$$\tilde{L}_{nm} \equiv k^2 \left[+ \sum_{j=1}^n \tilde{\epsilon}(x, \bar{\rho}_j) - \sum_{l=1+n}^{n+m} \tilde{\epsilon}^*(x, \bar{\rho}_l) \right] \quad (27)$$

Two related palatable approximations must now be made; since $L_{nm} \equiv \langle L_{nm} \rangle + \tilde{L}_{nm}$, and it is usually assumed that $|\tilde{L}_{nm}| \ll 1$, one has

$$L_{nm}^{-1} = [\langle L_{nm} \rangle + \tilde{L}_{nm}]^{-1} \approx \langle L_{nm} \rangle^{-1}, \quad [1 - \langle \tilde{L}_{nm} L_{nm}^{-1} \rangle]^{-1} \approx 1 \quad (28)$$

where

$$\begin{aligned} \langle L_{nm} \rangle^{-1} &= \left[2ik \frac{\partial}{\partial x} + \sum_{j=1}^n \nabla_{\rho_j}^2 - \sum_{l=1+n}^{n+m} \nabla_{\rho_l}^2 \right]^{-1} \\ &\approx \left[2ik \frac{\partial}{\partial x} \right]^{-1} \\ &= \frac{1}{2ik} \int_0^x dx' \end{aligned} \quad (29)$$

Equation (25) then becomes

$$\left[2ik \frac{\partial}{\partial x} + \sum_{j=1}^n \nabla_{\rho_j}^2 - \sum_{l=1+n}^{n+m} \nabla_{\rho_l}^2 - \langle \tilde{L}_{nm} \langle L_{nm}^{-1} \rangle \tilde{L}_{nm} \rangle \right] \Gamma_{nm} = 0 \quad (30)$$

where

$$\begin{aligned} \langle \tilde{L}_{nm} \langle L_{nm}^{-1} \rangle \tilde{L}_{nm} \rangle &\approx \frac{k^3}{2i} \int_0^x \left[\sum_{i=1}^n \sum_{k=1}^n \langle \tilde{\epsilon}(x, \bar{\rho}_i) \tilde{\epsilon}(x', \bar{\rho}'_k) \rangle - \sum_{i=1}^n \sum_{l=1+n}^{n+m} \langle \tilde{\epsilon}(x, \bar{\rho}_i) \tilde{\epsilon}^*(x', \bar{\rho}'_l) \rangle \right. \\ &\quad \left. - \sum_{j=1+n}^{n+m} \sum_{k=1}^n \langle \tilde{\epsilon}^*(x, \bar{\rho}_j) \tilde{\epsilon}(x', \bar{\rho}'_k) \rangle + \sum_{j=1+n}^{n+m} \sum_{l=1+n}^{n+m} \langle \tilde{\epsilon}^*(x, \bar{\rho}_j) \tilde{\epsilon}^*(x', \bar{\rho}'_l) \rangle \right] dx' \end{aligned} \quad (31)$$

which is the well-known classical paraxial form for the problem (ref. 13). It is interesting to note that the ‘geometrical optics’ approximation made in equation (29) leads to the parabolic equation in the paraxial approximation. Thus, one can envision a substantial extension of this development beyond that of the classical treatment if one is to use the entire form of the operator $\langle L_{nm} \rangle^{-1}$, i.e., use the inverse of the operator $\langle L_{nm} \rangle$ as solved in the paraxial approximation rather than in the geometrical optics approximation which was used above. This will form the subject of the next section in which the first order statistical moment and the generalized second moment (i.e., the mutual coherence function) are derived using the above formalism.

3.2 Solution for the First Moment

The first order moment of the random electric field in a plane transverse to the direction of propagation is defined through equation (11) to be given by

$$\Gamma_{10}(x; \vec{\rho}_1) = \langle E(x; \vec{\rho}_1) \rangle \quad (32)$$

which is a solution of the operator relation of equation (25), viz,

$$\left\{ \langle L_{10} \rangle - \left[1 - \langle \tilde{L}_{10} L_{10}^{-1} \rangle \right]^{-1} \langle \tilde{L}_{10} L_{10}^{-1} \tilde{L}_{10} \rangle \right\} \Gamma_{10} = 0 \quad (33)$$

where, from equations (15) and (16),

$$\langle \hat{L}_{10} \rangle \equiv 2ik \frac{\partial}{\partial x} + U_\rho, \tilde{L}_{10} \equiv 2k^4 V_{\rho_1} \tilde{\epsilon}(x, \vec{\rho}), \vec{\rho} \equiv \vec{\rho}_1 \quad (34)$$

The ability to proceed in an analytical fashion is dependent upon the simplifying approximations used earlier, viz, one has that

$$L_{10}^{-1} = \left[\langle L_{10} \rangle + \tilde{L}_{10} \right]^{-1} \approx \langle L_{10} \rangle^{-1} \left[1 - \langle \tilde{L}_{10} L_{10}^{-1} \rangle \right]^{-1} \approx 1 \quad (35)$$

Using this result, equation (33) becomes

$$\left\{ \langle L_{10} \rangle - \langle \tilde{L}_{10} \langle L_{10} \rangle^{-1} \tilde{L}_{10} \rangle \right\} \Gamma_{10} = 0 \quad (36)$$

Employing the appropriate definitions of the operators, equation (36) gives

$$\left[2ik \frac{\partial}{\partial x} + U_\rho - \left\langle \left(2k^4 V_\rho \tilde{\epsilon}(x, \vec{\rho}) \right) \left\{ 2ik \frac{\partial}{\partial x} + U_\rho \right\}^{-1} \left(2k^4 V_\rho \tilde{\epsilon}(x, \vec{\rho}) \right) \right\rangle \right] \Gamma_{10}(x, \vec{\rho}) = 0 \quad (37)$$

This differential equation in the operators U_ρ and V_ρ must now be simplified and solved for the first-order moment $\Gamma_{10}(x, \vec{\rho})$.

To this end, one must first deal with the factor

$$\left\{ 2ik \frac{\partial}{\partial x} + U_\rho \right\}^{-1} \equiv G(x, \vec{\rho}) \quad (38)$$

which is the Green function of the operators $2ik \partial/\partial x + \hat{U}_\rho$, defined by

$$\left\{ 2ik \frac{\partial}{\partial x} + 2k^2 \left(1 + \frac{\nabla_\rho^2}{k^2} \right)^{1/2} \right\} G(x, \vec{\rho}) = \delta(x - x') \delta(\vec{\rho} - \vec{\rho}') \quad (39)$$

where the definition of U_ρ is used. Applying the approximation $U_\rho \approx 2k^2 + \nabla_\rho^2$ and solving for the green function $G(x, \bar{\rho})$ yields,

$$G(x, \bar{\rho}) = G(x, \bar{\rho}; x', \bar{\rho}') = \left(\frac{1}{4\pi} \right) \exp[-ik(x - x')] \frac{\exp[-ik(\bar{\rho} - \bar{\rho}')^2 / 2(x - x')]}{x - x'} \quad (40)$$

Thus, the third term within the brackets of equation (37) can be written

$$\begin{aligned} \langle \dots \rangle &\equiv \left\langle \left(2k^4 V_\rho \tilde{\epsilon}(x, \bar{\rho}) \right) \left\{ 2ik \frac{\partial}{\partial x} + U_\rho \right\}^{-1} \left(2k^4 V_\rho \tilde{\epsilon}(x, \bar{\rho}) \right) \right\rangle = \\ &= (2k^4)^2 \int_0^x \int_{-\infty}^{\infty} G(x, \bar{\rho}; x', \bar{\rho}') \langle V_\rho \tilde{\epsilon}(x, \bar{\rho}) V_{\rho'} \tilde{\epsilon}(x, \bar{\rho}') \rangle d^2 \rho' dx' \end{aligned} \quad (41)$$

Proceeding further, one now must deal with the operator products

$$V_\rho \tilde{\epsilon}(x, \bar{\rho}) = \left(\frac{1}{2k^2} \right) \left(1 + \frac{\nabla_\rho^2}{k^2} \right)^{-1/2} \tilde{\epsilon}(x, \bar{\rho}) \quad (42)$$

Since $\tilde{\epsilon}(x, \bar{\rho})$ is a random function, it can be represented in the form of a Fourier-Stieltjes integral (ref. 16), i.e.,

$$\tilde{\epsilon}(x, \bar{\rho}) = \int \exp(i\bar{\kappa} \cdot \bar{\rho}) dZ(x, \bar{\kappa}) \quad (43)$$

in which the spectral amplitude $dZ(x, \bar{\rho})$ is endowed with the same statistical properties as is the random function $\tilde{\epsilon}(x, \bar{\rho})$ as will be shown in what is to follow. Applying equation (43) to equation (42) gives

$$V_\rho \tilde{\epsilon}(x, \bar{\rho}) = \left(\frac{1}{2k^2} \right) \int \left(1 - \frac{\kappa^2}{k^2} \right)^{-1/2} \exp(i\bar{\kappa} \cdot \bar{\rho}) dZ(x, \bar{\kappa}) \quad (44)$$

Thus, the ensemble averaged product appearing in right side of equation (41) becomes,

$$\begin{aligned} \langle V_\rho \tilde{\epsilon}(x, \bar{\rho}) V_{\rho'} \tilde{\epsilon}(x', \bar{\rho}') \rangle &= \left(\frac{1}{2k^2} \right)^2 \iint \left(1 - \frac{\kappa^2}{k^2} \right)^{-1/2} \left(1 - \frac{\kappa'^2}{k'^2} \right)^{-1/2} \\ &\cdot \exp(i\bar{\kappa} \cdot \bar{\rho} + i\bar{\kappa}' \cdot \bar{\rho}') \langle dZ(x, \bar{\kappa}) dZ(x', \bar{\kappa}') \rangle \end{aligned} \quad (45)$$

One now makes use of the fact that the atmospheric permittivity fluctuation field $\tilde{\epsilon}(x, \bar{\rho})$ is taken to be statistically homogeneous, characterized by a power spectral density $\Phi_\epsilon(x, \bar{\kappa})$ in the transverse plane, and δ -correlated in the longitudinal direction; these circumstances allow one to write (ref. 16)

$$\langle dZ(x, \bar{\kappa}) dZ(x', \bar{\kappa}') \rangle = \delta(\bar{\kappa} + \bar{\kappa}') F_\epsilon(x - x', \bar{\kappa}) d^2 \kappa d^2 \kappa' \quad (46a)$$

where for δ -correlated fluctuations in the x direction, the two-dimensional spectrum $F_\varepsilon(x - x', \vec{\kappa})$ is given by

$$F_\varepsilon(x - x', \vec{\kappa}) = 2\pi\delta(x - x')\Phi_\varepsilon(\vec{\kappa}) \quad (46b)$$

in which $\Phi(\vec{\kappa})$ is the three-dimensional spectrum of permittivity fluctuations. Using these relations in equation (45) and performing the integrations where possible yields

$$\langle V_{\vec{\rho}} \tilde{\varepsilon}(x, \vec{\rho}) V_{\vec{\rho}'} \tilde{\varepsilon}(x', \vec{\rho}') \rangle = 2\pi \left(\frac{1}{2k^2} \right)^2 \delta(x - x') \int_{-\infty}^{\infty} \left(1 - \frac{\kappa^2}{k^2} \right)^{-1} \exp[i\vec{\kappa} \cdot \vec{\rho}_d] \Phi_\varepsilon(\vec{\kappa}) d^2\kappa \quad (47)$$

where $\vec{\rho}_d \equiv \vec{\rho} - \vec{\rho}'$ is the difference coordinate.

Taking the statistics governing the random field $\tilde{\varepsilon}(x, \vec{\rho})$ to be also isotropic, i.e., $\Phi_\varepsilon(x, \vec{\kappa}) = \Phi_\varepsilon(x, \kappa)$, equation (41) can now finally be evaluated by substituting into it equations (40) and (47); converting the integration in the $\vec{\rho}_d$ -plane into one in plane polar coordinates and performing the associated integrations gives

$$\langle \dots \rangle = -i\pi^2 k^3 \int_0^\infty \left(1 - \frac{\kappa^2}{k^2} \right)^{-1} \Phi_\varepsilon(\kappa) \kappa d\kappa \quad (48)$$

where the δ -function relation

$$\int_0^x \delta(x - x') dx' = \frac{1}{2} \quad (49)$$

is employed.

Returning to equation (37) and, substituting equation (48) into equation (37) gives

$$\left[2ik \frac{\partial}{\partial x} + 2k^2 \left(1 + \frac{\nabla_{\vec{\rho}}^2}{k^2} \right)^{1/2} + i\pi^2 k^3 \int_0^\infty \left(1 - \frac{\kappa^2}{k^2} \right)^{-1} \Phi_\varepsilon(\kappa) \kappa d\kappa \right] \Gamma_{10}(x, \vec{\rho}) = 0 \quad (50)$$

In the plane-wave case, one has that

$$\left(1 + \frac{\nabla_{\vec{\rho}}^2}{k^2} \right)^{1/2} \Gamma_{10}(x, \vec{\rho}) = \left(1 + \frac{\nabla_{\vec{\rho}}^2}{k^2} \right)^{1/2} \Gamma_{10}(x) = \Gamma_{10}(x) \quad (51)$$

since the plane wave will not possess any transverse variations. In this special case, equation (50) becomes

$$\left[2ik \frac{\partial}{\partial x} + 2k^2 + i\pi^2 k^3 \int_0^\infty \left(1 - \frac{\kappa^2}{k^2} \right)^{-1} \Phi_\varepsilon(\kappa) \kappa d\kappa \right] \Gamma_{10}(x) = 0 \quad (52)$$

the solution of which is

$$\Gamma_{10}(x) = \Gamma_{10}(0) \exp \left[ikx - \frac{\pi^2}{2} k^2 x \int_0^\infty \left(1 - \frac{\kappa^2}{k^2} \right)^{-1} \Phi_\varepsilon(\kappa) \kappa d\kappa \right] \quad (53)$$

This result differs by the factor $(1 - \kappa^2/k^2)^{-1}$ from that of the parabolic equation in the paraxial approximation. Of course, the later is obtained from the former by retaining the first term in the series expansion of the factor. The presence of this factor tends to accentuate the spatial frequencies near the value of the wave number k . Since this theory is applicable to those situations in which $\lambda < l_0$, most of the contribution of this factor to wave scattering will occur at the largest spatial frequencies of the inhomogeneities. Hence, in the case of atmospheric turbulence, one can employ the von Karman spectral density

$$\Phi_\varepsilon(\kappa) = 0.033 C_\varepsilon^2 (\kappa^2 + K_0^2)^{-11/6} \quad (54)$$

which is not bounded at the high spatial frequencies, to compare the result of equation (53) with that of the paraxial approximation. Substituting equation (54) into equation (53) and evaluating the integral (ref. 13) yields

$$\Gamma_{10}(x) = \Gamma_{10}(0) \exp \left[ikx - \left\{ (1.772 + 1.023i) k^{-5/3} + 0.3908 K_0^{-5/3} {}_2F_1 \left[1, 1; \frac{1}{6}; -\frac{K_0^2}{k^2} \right] \right\} C_\varepsilon^2 k^2 x \right] \quad (55)$$

where the structure parameters for the permittivity and refractive index are related by $C_\varepsilon^2 = 4C_n^2$. In the case of the open atmosphere, one always has within the bounds of the extended parabolic equation, $k \gg K_0$; thus, the hypergeometric function reduces to unity and $k^{-5/3} \ll K_0^{-5/3}$, allowing equation (55) to be approximated by

$$\Gamma_{10}(x) = \Gamma_{10}(0) \exp \left[ikx - 0.3908 K_0^{-5/3} C_\varepsilon^2 k^2 x \right] \quad (56)$$

which is the result of the parabolic equation in the paraxial approximation. Hence, the use of the extended parabolic equation only makes negligible amplitude and phase corrections to the first order moment of the wave field. This result establishes the accuracy of the parabolic equation in the paraxial approximation for the first order moment (mean field) as it applies to atmospheric turbulence. The next section will consider the calculation of the generalized second-order moment (generalized mutual coherence function) of the wave field from the extended parabolic equation and compare its result to that of the paraxial approximation. In addition, due to the completeness of the operator analysis, one naturally obtains expressions for the MCF along the longitudinal axis.

3.3 Solution for the Generalized MCF (second moment)

The generalized MCF of the random electric field in two planes transverse to the direction of propagation is, from equation (11), given by

$$\Gamma_{11}(x_1, \bar{\rho}_1; x_2, \bar{\rho}_2) \equiv \langle E(x_1; \bar{\rho}_1) E^*(x_2; \bar{\rho}_2) \rangle \quad (57)$$

which is a solution of equation (25), in this case given by

$$\left\{ \langle L_{11} \rangle - \left[1 - \langle \tilde{L}_{11} L_{11}^{-1} \rangle \right]^{-1} \langle \tilde{L}_{11} L_{11}^{-1} \tilde{L}_{11} \rangle \right\} \Gamma_{11} = 0 \quad (58)$$

where, from the definitions of equations (15) and (16),

$$\langle L_{11} \rangle \equiv 2ik \frac{\partial}{\partial x_1} + 2ik \frac{\partial}{\partial x_2} + U_{\rho_1} - U_{\rho_2}^* \quad (59)$$

and

$$\tilde{L}_{11} \equiv 2k^4 \left[V_{\rho_1} \tilde{\epsilon}(x_1, \bar{\rho}_1) - V_{\rho_2}^* \tilde{\epsilon}^*(x_2, \bar{\rho}_2) \right] \quad \langle \tilde{L}_{11} \rangle = 0 \quad (60)$$

As with the case for the first order moment, two related approximations must be made at the outset to render the problem analytically tractable. In particular, so long as $|\tilde{L}_{11}| \ll 1$,

$$L_{11}^{-1} = \left[\langle L_{11} \rangle + \tilde{L}_{11} \right]^{-1} \approx \langle L_{11} \rangle^{-1}, \quad \left[1 - \langle \tilde{L}_{11} L_{11}^{-1} \rangle \right]^{-1} \approx 1 \quad (61)$$

which allows equation (58) to become

$$\left\{ \langle L_{11} \rangle - \langle \tilde{L}_{11} \langle L_{11} \rangle^{-1} \tilde{L}_{11} \rangle \right\} \Gamma_{11} = 0 \quad (62)$$

At this point, it is suggested to connect the longitudinal coordinates x_1 and x_2 to the related centroid x_c and difference x_d coordinates,

$$x_c \equiv \frac{x_1 + x_2}{2}, \quad x_d \equiv x_1 - x_2 \quad (63)$$

the operator expressions of equations (59) and (60) then become

$$\langle L_{11} \rangle \equiv 2ik \frac{\partial}{\partial x_c} + U_{\rho_1} - U_{\rho_2}^* \quad (64)$$

$$\tilde{L}_{11} \equiv 2k^4 \left[V_{\rho_1} \tilde{\epsilon} \left(x_c + \frac{x_d}{2}, \bar{\rho}_1 \right) - V_{\rho_2}^* \tilde{\epsilon}^* \left(x_c - \frac{x_d}{2}, \bar{\rho}_2 \right) \right], \quad \langle \tilde{L}_{11} \rangle = 0 \quad (65)$$

Hence, equation (62) can be written as

$$\left[2ik \frac{\partial}{\partial x_c} + U_{\rho_1} - U_{\rho_2}^* - \left\langle \left(2k^4 \left\{ V_{\rho_1} \tilde{\epsilon} \left(x_c + \frac{x_d}{2}, \bar{\rho}_1 \right) - V_{\rho_2}^* \tilde{\epsilon}^* \left(x_c - \frac{x_d}{2}, \bar{\rho}_2 \right) \right\} \right) \right. \right. \\ \left. \left. \cdot \left\{ 2ik \frac{\partial}{\partial x_c} + U_{\rho_1} - U_{\rho_2}^* \right\}^{-1} \left(2k^4 \left\{ V_{\rho_1} \tilde{\epsilon} \left(x'_c + \frac{x_d}{2}, \bar{\rho}_1 \right) - V_{\rho_2}^* \tilde{\epsilon}^* \left(x'_c - \frac{x_d}{2}, \bar{\rho}_2 \right) \right\} \right) \right\rangle \right] \Gamma_{11} = 0 \quad (66)$$

The solution of this equation commences with obtaining an expression for the Green function

$$\left\{ 2ik \frac{\partial}{\partial x_c} + U_{\rho_1} - U_{\rho_2}^* \right\}^{-1} \equiv G(x_c, \bar{\rho}_1, \bar{\rho}_2) \quad (67)$$

Proceeding as in the last section and using the approximation $\hat{U}_\rho \approx 2k^2 + \nabla_\rho^2$, this requires the solution of

$$\left\{ 2ik \frac{\partial}{\partial x_c} + U_{\rho_1} - U_{\rho_2}^* \right\} G(x, \bar{\rho}_1, \bar{\rho}_2) = \delta(x_c - x'_c) \delta(\bar{\rho}_1 - \bar{\rho}'_1) \delta(\bar{\rho}_2 - \bar{\rho}'_2) \quad (68)$$

which is given by

$$\begin{aligned} G(x_c, \bar{\rho}_1, \bar{\rho}_2) &= -\left(\frac{ik}{2}\right) \left(\frac{1}{2\pi}\right)^2 \left(\frac{1}{x_c - x'_c}\right)^2 \exp[ik((\bar{\rho}_1 - \bar{\rho}'_1) - (\bar{\rho}_2 - \bar{\rho}'_2))/2(x_c - x'_c)] \\ &= G(x_c - x'_c, \bar{\rho}_1 - \bar{\rho}'_1, \bar{\rho}_2 - \bar{\rho}'_2) \end{aligned} \quad (69)$$

Therefore, the fourth term in equation (66) is given by

$$\begin{aligned} \langle \dots \rangle &= (2k^4)^2 \int_0^x \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, \bar{\rho}_1, \bar{\rho}_2; x', \bar{\rho}'_1, \bar{\rho}'_2) \left(V_{\rho_1} V_{\rho'_1} \tilde{\varepsilon}^* \left(x_c - \frac{x_d}{2}, \bar{\rho}_1 \right) \tilde{\varepsilon} \left(x'_c + \frac{x_d}{2}, \bar{\rho}'_1 \right) - \right. \\ &\quad \left. - V_{\rho_1} V_{\rho'_2}^* \tilde{\varepsilon} \left(x_c + \frac{x_d}{2}, \bar{\rho}_1 \right) \tilde{\varepsilon}^* \left(x'_c - \frac{x_d}{2}, \bar{\rho}'_2 \right) - V_{\rho_2}^* V_{\rho'_1} \tilde{\varepsilon}^* \left(x_c - \frac{x_d}{2}, \bar{\rho}_2 \right) \tilde{\varepsilon} \left(x'_c + \frac{x_d}{2}, \bar{\rho}'_1 \right) + \right. \\ &\quad \left. + V_{\rho_2}^* V_{\rho'_2}^* \tilde{\varepsilon}^* \left(x_c - \frac{x_d}{2}, \bar{\rho}_2 \right) \tilde{\varepsilon}^* \left(x'_c - \frac{x_d}{2}, \bar{\rho}'_2 \right) \right) d^2 \rho'_1 d^2 \rho'_2 dx'_c \end{aligned} \quad (70)$$

One now employs the Fourier-Stieltjes transform as before to represent the products of $V_\rho \varepsilon$, i.e.,

$$V_\rho \tilde{\varepsilon} \left(x_c \pm \frac{x_d}{2}, \bar{\rho}_{1,2} \right) = \frac{1}{2k^2} \int \left(1 - \frac{\kappa_{1,2}^2}{k^2} \right)^{-1/2} \exp(i\bar{\kappa}_{1,2} \cdot \bar{\rho}_{1,2}) dZ \left(x_c \pm \frac{x_d}{2}, \bar{\kappa}_{1,2} \right), \text{ etc.}, \quad (71)$$

and obtains the following relations (functional arguments have been suppressed but the correspondence to those in equation (70) follows)

$$\langle V_{\rho_1} V_{\rho'_1} \tilde{\varepsilon} \tilde{\varepsilon} \rangle = \left(\frac{1}{2k^2} \right)^2 \int \left(1 - \frac{\kappa_1^2}{k^2} \right)^{-1} \exp[i\bar{\kappa}_1 \cdot (\bar{\rho}_1 - \bar{\rho}'_1)] F_\varepsilon(x_c - x'_c, \bar{\kappa}_1) d^2 \kappa_1 \quad (72a)$$

$$\langle V_{\rho_1} V_{\rho'_2}^* \tilde{\varepsilon} \tilde{\varepsilon}^* \rangle = \left(\frac{1}{2k^2} \right)^2 \int \left(1 - \frac{\kappa_1^2}{k^2} \right)^{-1} \exp[i\bar{\kappa}_1 \cdot (\bar{\rho}_1 - \bar{\rho}'_2)] F_\varepsilon(x_c - x'_c + x_d, \bar{\kappa}_1) d^2 \kappa_1 \quad (72b)$$

$$\langle V_{\rho_2}^* V_{\rho_1} \tilde{\varepsilon}^* \tilde{\varepsilon} \rangle = \left(\frac{1}{2k^2} \right)^2 \int \left(1 - \frac{\kappa_2^2}{k^2} \right)^{-1} \exp[-i\bar{\kappa}_2 \cdot (\bar{\rho}_2 - \bar{\rho}_1')] F_\varepsilon(x_c - x'_c - x_d, \bar{\kappa}_2) d^2 \kappa_2 \quad (72c)$$

$$\langle V_{\rho_2}^* V_{\rho_2}^* \tilde{\varepsilon}^* \tilde{\varepsilon}^* \rangle = \left(\frac{1}{2k^2} \right)^2 \int \left(1 - \frac{\kappa_2^2}{k^2} \right)^{-1} \exp[-i\bar{\kappa}_2 \cdot (\bar{\rho}_2 - \bar{\rho}_2')] F_\varepsilon(x_c - x'_c, \bar{\kappa}_2) d^2 \kappa_2 \quad (72d)$$

Substituting equations (69) and (72a) to (72d) into equation (70), performing the required integrals over $\bar{\rho}_1'$ and $\bar{\rho}_2'$ in plane polar coordinates, and taking all the spectra $F_\varepsilon(x, \bar{\kappa}_{1,2}) = F_\varepsilon(x, \kappa_{1,2})$, i.e., to be isotropic in the frequency $\bar{\kappa}$, one obtains

$$\begin{aligned} \langle \dots \rangle = & -i\pi k^3 \int_0^{x_c} \int_0^\infty \left(1 - \frac{\kappa^2}{k^2} \right)^{-1} \left\{ \exp \left[-i \frac{\kappa^2 (x_c - x'_c)}{2k} \right] F_\varepsilon(x_c - x'_c, \kappa) - \right. \\ & - \exp \left[i \frac{\kappa^2 (x_c - x'_c)}{2k} \right] J_0(\kappa \rho_d) F_\varepsilon(x_c - x'_c + x_d, \kappa) - \\ & - \exp \left[-i \frac{\kappa^2 (x_c - x'_c)}{2k} \right] J_0(\kappa \rho_d) F_\varepsilon(x_c - x'_c - x_d, \kappa) + \\ & \left. + \exp \left[i \frac{\kappa^2 (x_c - x'_c)}{2k} \right] J_0(\kappa \rho_d) F_\varepsilon(x_c - x'_c, \kappa) \right\} \kappa d\kappa dx_c \end{aligned} \quad (73)$$

where $\bar{\rho}_d \equiv \bar{\rho}_1 - \bar{\rho}_2$ is the difference coordinate. Using equation (46b) in equation (73) and performing the x_c integration, remembering equation (49), finally gives

$$\langle \dots \rangle = -2\pi^2 i k^3 \int_0^\infty \left(1 - \frac{\kappa^2}{k^2} \right)^{-1} \left\{ 1 - \exp \left(-\frac{i\kappa^2 x_d}{2k} \right) J_0(\kappa \rho_d) \right\} \Phi_\varepsilon(\kappa) \kappa d\kappa \quad (74)$$

Hence, equation (66) becomes

$$\left[2ik \frac{\partial}{\partial x_c} + U_{\rho_1} - U_{\rho_2}^* + 2\pi^2 i k^3 \int_0^\infty \left(1 - \frac{\kappa^2}{k^2} \right)^{-1} \left\{ 1 - \exp \left(-\frac{i\kappa^2 x_d}{2k} \right) J_0(\kappa \rho_d) \right\} \Phi_\varepsilon(\kappa) \kappa d\kappa \right] \Gamma_{11} = 0 \quad (75)$$

where $\Gamma_{11} = \Gamma_{11}(x_c, x_d, \rho_d)$. Making the plane wave approximation of equation (75), analogous to that done earlier for equation (50), the resulting differential equation has as a solution

$$\Gamma_{11}(x_c, x_d, \rho_d) = \Gamma_{11}(0, 0, \rho_d) \exp \left[-\pi^2 k^2 x_c \int_0^\infty \left(1 - \frac{\kappa^2}{k^2} \right)^{-1} \left\{ 1 - \exp \left(-\frac{i\kappa^2 x_d}{2k} \right) J_0(\kappa \rho_d) \right\} \Phi_\varepsilon(\kappa) \kappa d\kappa \right] \quad (76)$$

(More explicitly, one has in equation (75))

$$\begin{aligned}
(U_{\rho_1} - U_{\rho_2}^*) \Gamma_{11}(x_1, \bar{\rho}_1; x_2, \bar{\rho}_2) &\approx (\bar{\nabla}_{\rho_1}^2 - \bar{\nabla}_{\rho_2}^2) \Gamma_{11}(x_1, \bar{\rho}_1; x_2, \bar{\rho}_2) \\
&= 2\bar{\nabla}_R \cdot \bar{\nabla}_{\rho_d} \Gamma_{11}(x_1, x_2, \bar{\rho}_d) \\
&= 0
\end{aligned}$$

where $\bar{R} \equiv (\bar{\rho}_1 + \bar{\rho}_2)/2$ is the centroid coordinate.) It is important to note the initial condition $\Gamma(0, 0, \rho_d)$; since one necessarily must take x_c , one must also have $x_d = 0$ by equation (63) since $x_1, x_2 \geq 0$.

This result cannot be analytically studied in its entirety and hence will be considered in two special cases. The first case is defined by $x_d = 0$ in which one deals with the transverse MCF $\Gamma(x_c, \rho_d) \equiv \Gamma(x_c, 0, \rho_d)$. Thus, the extended parabolic equation solution for the MCF in a transverse plane at a distance x_c from the source is, from equation (76)

$$\Gamma_{11}(x_c, \rho_d) = \Gamma_{11}(0, 0, \rho_d) \exp \left[-\pi^2 k^2 x_c \int_0^\infty \left(1 - \frac{\kappa^2}{k^2} \right)^{-1} \{1 - J_0(\kappa \rho_d)\} \Phi_\epsilon(\kappa) \kappa d\kappa \right] \quad (77)$$

As with the case of the first-order moment, this result differs from that of the paraxial approximation in the presence of the factor $(1 - \kappa^2/k^2)^{-1}$ which serves to accentuate spectral contributions for frequencies near k . The integral indicated in this expression cannot be analytically evaluated using the von Karman spectrum, equation (54). However, since the integrand is such that no singularities exist in the use of the unbounded Kolmogorov spectrum, viz., equation (54) with $K_0 = 0$, one can use such a spectrum in equation (77) and, upon evaluating the integral, obtain

$$\begin{aligned}
\Gamma_{11}(x_c, \rho_d) = \Gamma_{11}(0, \rho_d) \exp \left[- \left\{ (3.544 - 2.048i) k^{-5/3} (1 - J_0(k \rho_d)) + \right. \right. \\
\left. \left. + 0.108 k^2 \rho^{11/3} {}_1F_2 \left(1; \frac{17}{6}, \frac{17}{6}; -\frac{k^2 \rho_d^2}{4} \right) \right\} k^2 C_n^2 x_c \right] \quad (78)
\end{aligned}$$

where ${}_1F_2(\dots)$ is a generalized hypergeometric function and, as noted earlier, $C_\epsilon^2 = 4C_n^2$. At the outset, since $\lambda \ll \rho_d$ in most applications, the first term within the braces of equation (78), although quite interesting in structure, is negligible with respect to the second term. The hypergeometric function of the second term is most easily dealt with by first converting it to a Lommel function $s_{\mu,0}(\dots)$ (ref. 17, p. 986), i.e.,

$${}_1F_2 \left(1; \frac{17}{6}, \frac{17}{6}; -\frac{k^2 \rho_d^2}{4} \right) = \left(\frac{11}{3} \right)^2 (k \rho_d)^{-11/3} s_{8/3,0}(k \rho_d) \quad (79)$$

The Lommel function reduces (ref. 17, p. 985), in the case where $k \rho_d \gg 1$, to the simple approximate result $s_{8/3,0}(k \rho_d) \approx (k \rho_d)^{5/3}$. Using this in equation (79) and (78) becomes the paraxial result

$$\Gamma_{11}(x_c, \rho_d) = \Gamma_{11}(0, \rho_d) \exp \left[-1.457 k^2 C_n^2 \rho_d^{5/3} x_c \right]. \quad (80)$$

Again, as with the first-order moment, corrections to the second moment afforded by the extended parabolic equation are negligible in the case of atmospheric turbulence. This is due to the fact that the Kolmogorov spectral density level near the inner scale of turbulence is much smaller than it is at larger scale sizes. If another spectral density were considered, e.g., one describing propagation through an aerosol medium, this may not be the case and significant corrections to the transverse MCF may prevail.

The second special case in which equation (76) will be examined is in the instance where the factor $(1-\kappa^2/k^2)^{-1}$ can be neglected; as shown above, this is a good approximation for atmospheric turbulence. Thus, one is now dealing with the solution for the generalized MCF for field locations at different transverse and longitudinal points,

$$\Gamma_{11}(x_c, x_d, \rho_d) = \Gamma_{11}(0, 0, \rho_d) \exp \left[-\pi^2 k^2 x_c \int_0^\infty \left\{ 1 - \exp \left(-\frac{i\kappa^2 x_d}{2k} \right) J_0(\kappa \rho_d) \right\} \Phi_\varepsilon(\kappa) \kappa d\kappa \right] \quad (81)$$

Longitudinal correlations of the wave field have been previously considered in (refs. 18 and 19) using different methods and obtaining different results. Setting $x_d = 0$ in equation (81) gives the well known paraxial result for the transverse MCF $\Gamma(x_c, 0, \rho_d)$. Unlike the cases studied above, the form of equation (81) suggests the use of the Kolmogorov spectrum as modified by Tatarskii, which incorporates a cutoff at high spatial frequencies by allowing the introduction of the inner scale of turbulence l_0 , viz.,

$$\Phi_\varepsilon(\kappa) = 0.033 C_\varepsilon^2 \kappa^{-11/3} \exp \left(-\frac{\kappa^2}{\kappa_m^2} \right), \kappa_m \equiv \frac{5.92}{l_0} \quad (82)$$

Substituting equation (82) into equation (81) and evaluating the resulting integral yields for the generalized MCF

$$\Gamma_{11}(x_c, x_d, \rho_d) = \Gamma_{11}(0, 0, \rho_d) \exp \left[-4.352 k^2 C_\varepsilon^2 x_c \left\{ B^{5/6} {}_1F_1 \left(-\frac{5}{6}; 1; -\frac{\rho_d^2}{4B} \right) - \kappa_m^{-5/3} \right\} \right] \quad (83)$$

where

$$B \equiv \frac{1}{\kappa_m^2} + i \frac{x_d}{2k} \quad (84)$$

Thus, as first noted in (ref. 19), the presence of the diffraction factor on equation (81) modifies the effect of the cutoff frequency κ_m . In the case where $\rho_d > |1/\kappa_m^2 + i x_d/2k|^{1/2}$, equation (83) reduces to equation (80) upon employing the asymptotic representation of the confluent hypergeometric function ${}_1F_1(\dots)$. When $\rho_d = 0$, equation (83) becomes

$$\Gamma_{11}(x_c, x_d, 0) = \Gamma_{11}(0, 0, \rho_d) \exp \left[-4.352 k^2 C_\varepsilon^2 x_c \kappa_m^{-5/3} \left\{ \left(1 + \frac{i x_d \kappa_m^2}{2k} \right)^{5/6} - 1 \right\} \right] \quad (85)$$

In the appropriate limits, this expression gives

$$\Gamma_{11}(x_c, x_d, 0) = \Gamma_{11}(0, 0, \rho_d) \begin{cases} \exp\left[-1.8133ikC_n^2\kappa_m^{1/3}x_cx_d\right] & x_d\lambda \ll l_0^2 \\ \exp\left[-(0.5631 + 2.102i)k^{7/6}C_n^2x_cx_d^{5/6}\right] & x_d\lambda \gg l_0^2 \end{cases} \quad (86)$$

Hence, there is a phase variation in the longitudinal direction, as expected, with an attendant attenuation as the longitudinal separation is increased. It must be noted, however, that one needs to realize the condition $x_d \gg l_0$ in all cases so as to satisfy the assumption of δ -correlation of the fluctuations along the longitudinal axis (ref. 19). Figures 1 and 2 show plots of equation (85) with $\Gamma_{11}(0, 0, 0) = 1$ for a typical atmospheric propagation scenario at $\lambda = 0.63 \mu\text{m}$ for two values of inner scale size, $l_0 = 1.0 \text{ mm}$ and $l_0 = 1.0 \text{ cm}$, respectively.

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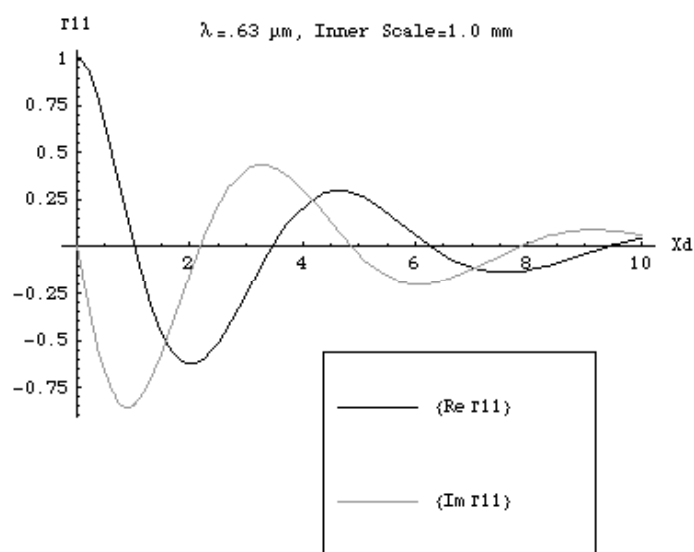


Figure 1.— $\Gamma_{11}(x_c, x_d, 0)$ versus x_d (in meters)
for $\lambda = 0.63$ mm, $l_0 = 1.0$ mm, $x_c = 5$ km,
and $C_n^2 = 10^{-12} \text{ m}^{-2/3}$

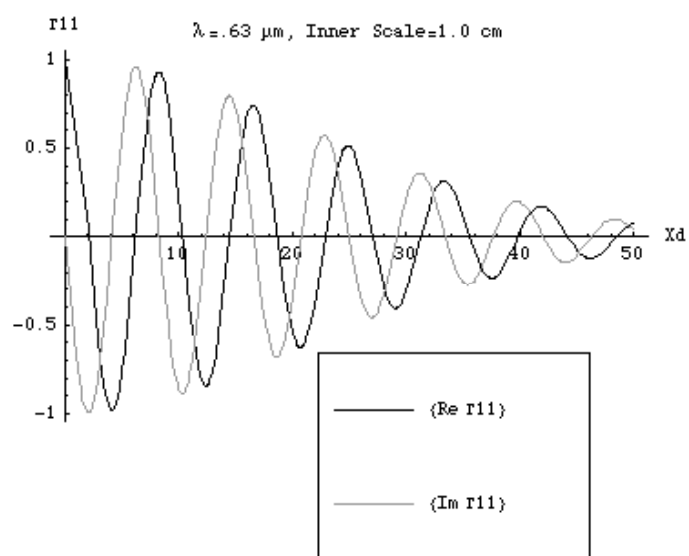


Figure 2.— $\Gamma_{11}(x_c, x_d, 0)$ versus x_d (in meters)
for $\lambda = 0.63$ mm, $l_0 = 1.0$ cm, $x_c = 5$ km,
and $C_n^2 = 10^{-12}$

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